

# Bipartite entanglement measure based on covariances

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(Dated: February 1, 2008)

We propose an entanglement measure for two quNits based on the covariances of a set of generators of the  $su(N)$  algebra. In particular, we represent this measure in terms of the mutually unbiased projectors for  $N$  prime. For pure states this measure quantifies entanglement, we obtain an explicit expression which relates it to the concurrence hierarchy, specifically the  $I$ -concurrence and the 3-concurrence. For mixed states we propose a separability criterion.

## I. INTRODUCTION

Entanglement plays a key role in quantum information and quantum communications processes. During the last years a wealth of entanglement measures have been proposed and studied [1]. In particular, the two-qubit case has been extensively studied, and entanglement of formation [2] and concurrence [3] are now widely accepted as entanglement measures. These measures require a complete knowledge of the density matrix, which, in turn, require state tomography, an experimentally and computationally labor intensive process.

Higher dimensional cases are more complicated. An accepted *separability criterion* is the so-called positive partial transpose (PPT) criterion [4], which is necessary and sufficient for composite systems with dimensions  $2 \times 2$  and  $2 \times 3$  [5], otherwise it is only necessary and it does not give information about the amount of entanglement. Motivated by the positive, but not completely positive maps, which are always positive for separable states [5], another important criterion has been introduced [6]. A separability criteria which identifies the entanglement in some states that PPT does not (so-called bound states), is the realignment method [7]. The method has the advantage that it gives a rough quantitative estimate of the degree of entanglement.

However, even for two qubits there is no consensus on how to *quantify* entanglement. Uhlmann introduced one measure that is based on the fact that antilinear operators are nonlocal [8]. Unfortunately, this generalization is not invariant under local unitary transformations, an important property that an entanglement measure requires. Rungta *et al.* introduced another generalization of concurrence [9], namely the  $I$ -concurrence, based on a generalization of the spin-flip operation called universal inverter. The measure possesses the requirements for a good entanglement measure [10], and theoretically is very nice, nevertheless the universal inverter is not a complete positive operation, so that it is not directly experimentally realizable. However, for bipartite systems with no more than two eigenvalues different from zero there is an explicit formula for the  $I$ -tangle, that is the square of the  $I$ -concurrence [11]. At roughly the same time, this concurrence was also introduced in [12] in terms of invariants under local unitary transformations.

For mixed states, the situation is further complicated. E.g., the  $I$ -concurrence [9] requires a global minimization over all bases which makes it cumbersome to calculate for mixed states. Mintert, Kuś, and Buchleitner [13] found a lower bound on  $I$ -concurrence which is simpler to estimate than the  $I$ -concurrence itself, and a short time later Chen, Alberverio and Fei found an analytical lower bound [14] connecting the  $I$ -concurrence with the PPT criterion [4], and the realignment criterion [7]. Another attempt of generalizing the concurrence [3] for mixed states in higher dimensions was made by Badziag *et al.* in [15]. They introduced the so called pre-concurrence, which, unfortunately, is difficult to analyze for states with a rank  $> 2$ . Moreover, there is no guarantee that the ensuing concurrence matrix can be diagonalized. Then, they introduce the biconcurrence, which implies a separability criteria, but it too, requires a minimization procedure. Yet another proposal to deal with mixed states in higher dimensions is presented in [16], where the concept of negativity is extended [17] for mixed states by means of a convex-roof, which gives a necessary and sufficient separability criteria. For two qubits it coincides with the concurrence [3]. Unfortunately, all these measures are difficult to implement experimentally and they require substantial efforts to estimate.

An easier way to detect entanglement is using entanglement witnesses [18]. Recently it was shown that with non-linear expressions, that often can be implemented experimentally without extra effort, any witness can be improved [19]. In fact, in [20], using the universal inverter [3, 5], a positive map that leads an optimal entanglement witness, in the sense that it can recognize more entanglement states with positive partial transpose than any other, is constructed. However entanglement witness needs to be tailor made for each quantum state. Hence *a priori* knowledge of the state is needed.

Several years ago Schlienz and Mahler proposed a general description of entanglement using the density matrix formalism [21]. For the bipartite case they introduce an entanglement tensor whose components are the covariances between a pair of generators of the respective algebra for each particle. They show that this tensor is the difference between the composite density matrix and the tensor product of the reduced density matrices for each subsystem. By taking the square form of this tensor one obtains a distance which is vanishing for any product

state and is positive otherwise. This distance is maximal for maximally entangled states, and it is invariant under local unitary transformations. The work in [21] focus on the entanglement of pure states, but they suggest that it should be possible to extend this result for mixed states. However, Schlienz and Mahler were ahead of their time because concurrence had not yet been proposed in 1995, so how to distinguish between entangled states and a statistical mixture of separable states is not discussed in [21].

The use of uncertainty relations in the study of entanglement is well known for continuous variable [22]. In [23] Hofmann and Takeuchi proposed a generalization the uncertainty principle to uncertainty sums of local observables for finite dimensional systems. They derived local uncertainty criterion valid for every bipartite separable state. This criterion was later extended to multiqubit systems and reformulated in a way that it can be connected with continuous variables thorough the covariance matrix [24]. Nevertheless, the local uncertainty sums depends on the sign of the covariances of the local observables, causing an unnatural asymmetry, and the range of nonseparable states that this local uncertainty relation are able to detect is small [23, 25]. The criterion was improved in [26], where with a slightly modification, the uncertainty relations can detect a larger class of nonseparable states with the same measurement data as in [23].

Recently, it was discovered that Schlienz and Mahler's measure [21] and local uncertainty relations are really just two sides of the same coin [27]. Schlienz and Mahler's measure can be stated as a criterion (a limit) to ensure the entanglement. For pure states the measure can be expressed in terms of the standard concurrence [3]. For a highly entangled state this measure can even quantify entanglement to some degree. Recently the measure was tested experimentally [28]. In Sec. II we extend the work of Schlienz and Mahler and Kothe and Björk on the separability limit and on the relation between the measure and entanglement invariants.

When trying to detect or quantify entanglement experimentally one needs to consider that quantum mechanics is based on probabilities. Hence, in order to obtain as much information as possible when measuring a quantum state not only a complete set of linearly independent measures are needed, but they should also optimize the process. Wootters and Fields [29] showed that measurements in mutually unbiased basis (MUB) provide a minimal and optimal way for a complete determination of a quantum state. The concept of mutual unbiasedness was introduced by Ivanović [30] who proved that for prime dimension such basis exist, by an explicit construction. Some time after this concept was extended for a power of prime dimensional spaces [31]. In Sec. III we combine the ideas introduced in [21], and in [27] with the idea of optimal experimental estimation of a state, or, in this case, specifically estimation of its entanglement.

## II. THE CORRELATION MEASURE FOR ARBITRARY DIMENSIONAL BIPARTITE SYSTEMS

In this section we extend the work on the correlation measure for two systems, made in [21, 27]. Specifically, we take the entanglement measure proposed in [21], and use it to prove a criterion for nonseparability and relate it to two entanglement invariants. An advantage with the criterion is that it is experimentally measurable, and it only involves correlations between local measurements.

Consider two systems  $A$  and  $B$  of dimensions  $N_A$  and  $N_B$  respectively, where we, without loss of generality, can assume that  $N_A \leq N_B$ . The generalization of the bipartite equation is straightforward [21],

$$G = \sum_{k=1}^{N_A^2-1} \sum_{l=1}^{N_B^2-1} |C(\hat{\lambda}_k^A, \hat{\lambda}_l^B)|^2, \quad (1)$$

where

$$C(\hat{\lambda}_k^A, \hat{\lambda}_l^B) = \langle \hat{\lambda}_k^A \otimes \hat{\lambda}_l^B \rangle - \langle \hat{\lambda}_k^A \otimes \hat{\mathbf{1}}^B \rangle \langle \hat{\mathbf{1}}^A \otimes \hat{\lambda}_l^B \rangle$$

is the covariance between  $\hat{\lambda}_k^A$  and  $\hat{\lambda}_l^B$  and where  $\hat{\lambda}_{k(l)}^{A(B)}$ ,  $k, l = 1, \dots, N_{A(B)}^2 - 1$  are the generators of the  $su(N_{A(B)})$  algebra. They fulfill the relations

$$\text{Tr}(\hat{\lambda}_k) = 0, \quad \text{Tr}(\hat{\lambda}_k \hat{\lambda}_l) = \delta_{kl}. \quad (2)$$

In two and tree dimensions a representation of these operators are the Pauli and the Gell-Mann matrices, respectively, that are listed in, e.g., [32]. For higher dimensions an explicit construction algorithm can be found in [33]. Note, however, that from an experimental point of view, some representations of  $su(N)$  groups are preferable over others. We will return to this point in Sec. III.

As was pointed out in [21], and later in [34] for the qubit case, the measure  $G$  is proportional the square of the Hilbert-Schmidt distance between the composite density matrix and the tensor product of the reduced density matrices,

$$G = \text{Tr} \{ (\hat{\rho} - \hat{\rho}^A \otimes \hat{\rho}^B)^2 \}, \quad (3)$$

where  $\hat{\rho}^{A(B)}$  is the reduced density matrix for the subsystem  $A(B)$ , and  $\hat{\rho}$  is that of the composite system. The density matrices for each subsystem can be written in terms any set of  $su(N)$  generators (see for example [33]), that is,

$$\hat{\rho}^A = \sum_{j=0}^{N_A^2-1} a_j \hat{\lambda}_j^A, \quad \hat{\rho}^B = \sum_{j=0}^{N_B^2-1} b_j \hat{\lambda}_j^B, \quad (4)$$

where here and below, we have taken  $\lambda_0^{A(B)} = \hat{\mathbf{1}}$  and therefore  $a_0 = N_A^{-1}$  and  $b_0 = N_B^{-1}$ . Since the direct product of the basis states of the single particles serves

as a basis in the composite system, the density matrix for the total system can be written as,

$$\hat{\rho} = \sum_{k=0}^{N_A-1} \sum_{l=0}^{N_B-1} l_{kl} \hat{\lambda}_k^A \otimes \hat{\lambda}_l^B. \quad (5)$$

Note that for  $k = l = 0$ , i.e., the first term, we have  $l_{00} = (N_A N_B)^{-1}$  irrespective of  $\hat{\rho}$ .

The key to probe (3) is that tracing over one of the subsystems simply corresponds to choosing the zero component for the corresponding index [34], in our notation,

$$\begin{aligned} a_k &= \text{Tr}(\hat{\rho}^A \hat{\lambda}_k^A) = N_B l_{k0} \\ b_l &= \text{Tr}(\hat{\rho}^B \hat{\lambda}_l^B) = N_A l_{0l}. \end{aligned}$$

The measure given by (1), or equivalently (3) has some desirable features. One is that in the Hilbert-Schmidt distance form (3) is easy to manipulate theoretically. It is straightforward to see some important properties such that it is invariant under local unitary transformations. It is also quite obviously zero for pure, separable states. For the maximally entangled states, i.e.,

$$|\psi\rangle = \frac{1}{\sqrt{N_A}} \sum_{j=1}^{N_A} |jj\rangle, \quad (6)$$

$G$  obtains its maximum,  $(N_A^2 - 1)/N_A^2$ .

In order to analyze the properties of the proposed measure, we will take it in the form (3). Consider any pure state in the Schmidt decomposition:

$$|\psi\rangle = \sum_{j=1}^{N_A} e^{i\alpha_j} \sqrt{a_j} |\psi_j^A\rangle \otimes |\psi_j^B\rangle,$$

where  $a_j$ ,  $j = 1, \dots, N_A$  are real and nonnegative,  $a_1 + \dots + a_{N_A} = 1$ , and  $\langle \psi_i^A | \psi_j^A \rangle = \langle \psi_i^B | \psi_j^B \rangle = \delta_{ij}$ . Inserting this state into (3) one obtains,

$$G = \sum_{i=1}^{N_A} a_i^4 + 2 \sum_{\substack{i,j=1 \\ i < j}}^{N_A} a_i^2 a_j^2 - 2 \sum_{i=1}^{N_A} a_i^3 + \sum_{i=1}^{N_A} a_i^2 + 2 \sum_{\substack{i,j=1 \\ i < j}}^{N_A} a_i a_j. \quad (7)$$

Now consider the generalization of the concurrence for bipartite systems in higher dimensions [9], the so called  $I$ -concurrence  $\mathcal{C}_I$ , given by

$$\mathcal{C}_I^2 = 1 - \text{Tr} \{ (\hat{\rho}^A)^2 \}. \quad (8)$$

$I$ -concurrence is an entanglement monotone, that is, it does not increase on average under local operations and classical communication. In the  $N_A N_B$  dimensional case, the square of the  $I$ -concurrence (8) reads [9],

$$\mathcal{C}_I^2 = 1 - \sum_{i=1}^{N_A} a_i^2 = 2 \sum_{\substack{i,j=1 \\ i < j}}^{N_A} a_i a_j. \quad (9)$$

$I$ -concurrence, being only one number, can not make a distinction between some different kinds of entangled states [15, 35]. That is, states may have the same  $I$ -concurrence although they cannot be transformed one into the other using local operations and classical communication (LOCC). Nielsen [36] gives necessary and sufficient conditions for state transformation processes and in [35] a concurrence hierarchy is defined. We know from that work that one needs  $N_A - 1$  independent invariants under local unitary transformations in a  $N_A$ -level quantum system for a complete characterization of entanglement. In our case, complementing the concurrence (8), we will consider the 3-concurrence  $\mathcal{C}_3$ , another invariant under local unitary transformations that is related with the entanglement between the superposition-state triads, and which does not increase under LOCC [35],

$$\mathcal{C}_3 = \sum_{\substack{i,j,k=1 \\ i < j < k}}^N a_i a_j a_k. \quad (10)$$

Using (8) and (10), we can, after some algebra, obtain a relation between  $G$ ,  $\mathcal{C}_I$  and  $\mathcal{C}_3$  for pure states:

$$G = \mathcal{C}_I^4 + \mathcal{C}_I^2 - 6\mathcal{C}_3. \quad (11)$$

As we can see, the measure  $G$  is a function of two of the invariants of the  $N_A - 1$  necessary for a complete characterization of the entanglement [15, 35, 36].

Now, we will propose a separability criterion for  $N_A N_B$ -dimensional systems. The limit to ensure entanglement for two qubits is  $G > 1/4$ . Note that in Ref. [27] the derived limit is a factor 4 higher because of a different definition (by a factor of two) of the group generators (2). We shall show that this limit is independent of the bipartite system dimensionality.

A maximally correlated separable state has the form

$$\hat{\rho} = \sum_{j=1}^{N_A} p_j |\psi_j^A\rangle \langle \psi_j^A| \otimes |\psi_j^B\rangle \langle \psi_j^B|, \quad (12)$$

where  $\langle \psi_i^A | \psi_j^A \rangle = \langle \psi_i^B | \psi_j^B \rangle = \delta_{ij}$ . The reason the maximally correlated state must have this form is that only this form allows, by a proper local unitary transform, or equivalently, by a properly chosen measurement basis, to get distinctly correlated measurement outcomes. If detector  $j_A$  “clicks”, indicating that state  $|\psi_j^A\rangle$  was detected, this form guarantees that detector  $j_B$  will also click. Hence, the local measurement outcomes are completely correlated. Using the method of Lagrange multipliers, it is not hard to find that the maximal value of  $G$  for such state, with the restraint that all  $p_j$  are real, non-negative, and  $\sum_{j=1}^{N_A} p_j = 1$ , is  $1/4$ . The state achieving this maximum has the form

$$\hat{\rho} = \frac{1}{2} (|00\rangle \langle 00| + |11\rangle \langle 11|), \quad (13)$$

for any  $N_A$  and  $N_B$ , and it is clear that any local transformations will keep the state as an equal statistical mixture of two tensor products of locally orthogonal states.

Hence, the criterion

$$G > 1/4 \quad (14)$$

ensures that the state is nonseparable.

Corresponding to the bipartite qubit case, it is possible to derive a lower limit of  $G$  as a function of the state's I-concurrence and 3-concurrence, and this limit is given by (11). This expression provides the lower limit because a pure state has all its correlations in the entanglement, whereas mixed states can also have statistical correlations, as shown by the example with an unentangled state in Eq. (13).

However, in contrast to the bipartite qubit case it is difficult to derive an upper limit to  $G$  as a function of  $\mathcal{C}_I$  and  $\mathcal{C}_3$  as the latter is undefined for mixed states. We also lack a systematic way of parameterizing general bipartite states with a given I-concurrence, and therefore we cannot derive the function's maximum for a given  $\mathcal{C}_I$ , except that we know that  $G$ 's global maximum is  $1 - N_A^{-2}$  for the state given in (6). What is clear from numerical simulations, and which was shown to hold for the bipartite qubit case, is that when  $G$  is close to its maximal value, the range over which  $\mathcal{C}_I$  and  $\mathcal{C}_3$  can vary while preserving the value of  $G$  is very small. Hence, for highly entangled states  $G$  will pinpoint both  $\mathcal{C}_I$  and  $\mathcal{C}_3$  through (11) relatively well.

The criterion (14) is sufficient but not necessary. A simple example of the latter is the isotropic, two qutrit state

$$\hat{\rho} = \frac{1-\alpha}{9}\hat{\mathbf{1}} + \frac{\alpha}{3} \sum_{m,n=1}^3 |mm\rangle\langle nn|, \quad -\frac{1}{8} \leq \alpha \leq 1. \quad (15)$$

For this state we obtain  $G = 8\alpha^2/9$ , which implies that for  $\alpha < 3/4\sqrt{2} \approx 0.53$  our measure (14) cannot say anything about separability, but it is known that for  $\alpha > 1/4$  the state (15) is entangled [37].

### III. OPTIMAL MEASUREMENT ESTIMATES OF ENTANGLEMENT

The measure (11) yields the same value irrespective of the set of  $su(N)$  generators one uses, provided that they fulfil (2). However, from an experimental point of view it is desirable that the generators are unbiased. That is, the generators should be as “different” from each other as possible. In the two-qubit case it is natural to take the  $su(2)$  generators to be the Pauli matrices. These generators are all mutually unbiased in that the absolute value of the scalar product between any eigenvector of one generator and any eigenvector of any other generator equals  $2^{-1/2}$ . This is not true for the Gell-Mann matrices where the corresponding eigenstate overlap spans between 0 to 1.

Since, starting from a finite ensemble of identically prepared states, we are interested to measure local correlations as well as possible, we want to minimize the estimation error due to the probabilistic nature of quantum

measurements. This can be done if we can construct a set of  $su(N)$  operators that simultaneously constitute a mutually unbiased basis set MUB [29]. Unfortunately the constructions of such sets depend on the dimensionality of the space. The qubit space has already been discussed, and for odd prime and integer powers of odd and even prime dimensions, it is possible to find one more MUB than the space dimension, which is what we need.

Let us start with the qutrit case first, and generalize this later. For each qutrit [30, 38], there exist 4 MUB, with 3 projectors each,  $|\phi_{i,k}\rangle\langle\phi_{i,k}|$ , where the subindex  $k = 1, \dots, 4$  denotes the basis and the subindex  $i = 1, \dots, 3$  denotes the element of the basis.

A common way to construct the MUB is finding 4 unitary matrices (one of them is the identity), and then transforming the standard basis (projectors) with them in order to obtain the 4 projectors of the MUB (see for example [30, 32, 38]).

In [32] one can find the 8 generators of  $su(3)$ , which as a functions of the MUB projectors  $\hat{\rho}_{i,k} = |\phi_{i,k}\rangle\langle\phi_{i,k}|$  are given by

$$L_k = \sqrt{\frac{1}{6}}(2\rho_{1,k} - \rho_{2,k} - \rho_{3,k}) \quad \text{and} \quad (16)$$

$$\tilde{L}_k = \sqrt{\frac{1}{2}}(\rho_{3,k} - \rho_{2,k}). \quad (17)$$

For convenience we label the operators (16) and (17)

$$L_1 = \hat{\lambda}_1, \quad \tilde{L}_1 = \hat{\lambda}_2, \dots, \tilde{L}_4 = \hat{\lambda}_8,$$

Is easy to check that these operators are generators of the  $su(3)$  algebra, in other words they fulfill (2).

In the form (16) and (17) we have eight generators in terms of 12 projectors, so if we insert this formula in (1), it seems like that one should need 144 correlations. This is a chimera since for each basis

$$\sum_{i=1}^3 \rho_{i,k} = \hat{\mathbf{1}}, \quad \forall \quad k = 1, \dots, 4, \quad (18)$$

and substituting in (16) and then in (1), we obtain our measure in terms of the MUB projectors' covariances:

$$\begin{aligned} G = & \sum_{k,l=1}^4 \left( 4 \left[ \sum_{i,j=2}^3 C(\rho_{ik}^A, \rho_{jl}^B) \right]^2 \right. \\ & - \sum_{i=2}^3 \sum_{\substack{j,j'=2 \\ k' \neq l'}}^3 C(\rho_{ik}^A, \rho_{jk}^B) C(\rho_{i,k'}^A, \rho_{j,l'}^B) \\ & - 3 \sum_{i,i'=2}^3 C(\rho_{ik}^A, \rho_{j2}^B) C(\rho_{ik'}^A, \rho_{j3}^B) \\ & \left. - 3 \sum_{i,i'=2}^3 C(\rho_{i2}^A, \rho_{jk}^B) C(\rho_{i3}^A, \rho_{jk'}^B) \right). \quad (19) \end{aligned}$$

As before, this leaves us with sixty four correlations to measure for two qutrits.

Now let us generalize the result above to dimension  $N$ , where  $N = 2n + 1$  is an odd prime. Using the notation introduced in [32] the  $k$ -th group of operators is given by

$$L_{l,k} = \frac{1}{\sqrt{2(2n+1)}}(O_k^l + O_k^{2n-l+1}), \quad l = 1, \dots, n$$

$$\tilde{L}_{l,k} = \frac{i}{\sqrt{2(2n+1)}}(O_k^{2n-l+1} - O_k^l), \quad l = 1, \dots, n,$$

for  $k = 0, \dots, 2n + 1$ , where  $O_k = (AD^k)$  for  $k = 0, \dots, 2n$ , and  $O_{2n+1} = D$ .  $A$  is the cyclic permutation matrix and  $D$  is the diagonal matrix which elements are the powers of the  $N$ -th root of unity,  $\omega = e^{2\pi i/N}$ , that is  $D = \text{diag}\{1, \omega, \omega^2, \dots, \omega^{2N}\}$ .

We can use the spectral decomposition to obtain these operators in terms of the MUB projectors,

$$L_{l,k} = \sqrt{\frac{2}{2n+1}} \sum_{j=0}^{2n} \cos(2\pi l j / (2n+1)) \rho_{j,k} \quad (20)$$

$$\tilde{L}_{l,k} = \sqrt{\frac{2}{2n+1}} \sum_{j=0}^{2n} \sin(2\pi l j / (2n+1)) \rho_{j,k}, \quad (21)$$

where  $\rho_{j,k}$  is the  $j$ -th eigenprojector of the  $k$ -th MUB, with eigenvalue  $\omega^j$ .

Following the procedure made in for two qutrits, the results (1), (20) and (21), and the fact that the projector set is overcomplete, one can construct an entanglement measure similar to (19).

For the case when, e.g.,  $N_A = p^k$  is a power of a prime number, one can construct the generators of the  $su(N_A)$  algebra in a similar way, with the unitary matrices given,

by example in [29], [31]. On the other hand, when  $N_A$  is a composite number of at least two different prime numbers, the corresponding set of mutually unbiased bases are unknown. It is even not known if one can find  $N_A + 1$  mutually unbiased bases. The evidence at hands is negative, so for such systems the estimation process is likely to be less efficient.

## IV. CONCLUSIONS

In this paper we have extend the work made by Schlienz and Mahler [21] and Kothe and Björk [27], taking the entanglement measure proposed in [21], to bipartite states of any dimension. For pure states, it can quantify entanglement in a certain way, and we derived a relation between this measure, the I-concurrence and the 3-concurrence (two entanglement invariants). For mixed states, we established a limit sufficient, but not necessary, to ensure nonseparability (14).

Taking into account that one can determine in an optimal way all properties of a state measuring all combinations of local MUB eigenstate projections and the identity, we have also given the measure in terms of MUB eigenprojectors.

## V. ACKNOWLEDGEMENTS

This work was supported by the Swedish Foundation for International Cooperation in Research and Higher Education (STINT), the Swedish Research Council (VR), and the Swedish Foundation for Strategic Research (SSF).

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